Quantum Differential and Linear Cryptanalysis

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Motivation

What would be the impact of quantum computers on symmetric cryptography?

- Some physicists think they can build quantum computers
- NSA thinks we need quantum-resistant crypto (or do they?)

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Expected impact of quantum computers

- Some problems can be solved much faster with quantum computers
  - Up to exponential gains
  - But we don't expect to solve all NP problems

Impact on public-key cryptography

- RSA, DH, ECC broken by Shor's algorithm
  - Breaks factoring and discrete log in polynomial time
  - Large effort to develop quantum-resistant algorithms (*e.g.* NIST)

Impact on symmetric cryptography

- Exhaustive search of a k-bit key in time $2^{k/2}$ with Grover's algorithm
  - Common recommendation: double the key length (AES-256)
- Encryption modes are secure [Unruh & al, PQC'16]
- Authentication modes broken w/ superposition queries [Crypto '16]

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Overview of the talk

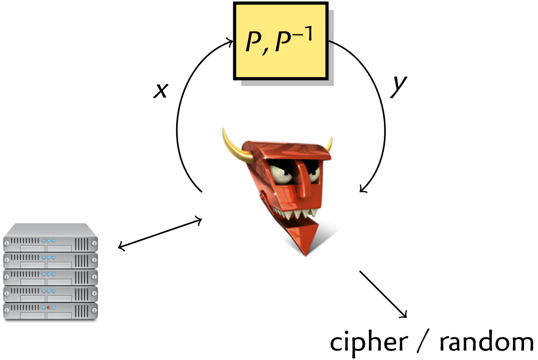
Main question

Is AES secure in a quantum setting?

- Symmetric design are evaluated with cryptanalysis:
  - Differential (truncated, impossible, ...)
  - Linear
  - Integral
  - Algebraic
  - ...
- We should study quantum cryptanalysis!
- Start with classical techniques
  - Do we get a quadratic speedup?
  - Do we need a quantum encryption oracle?
  - How are different cryptanalysis techniques affected?

Security notions: Classical

- PRF security: given access to P/P^{-1}, distinguishing E from random
- Classical setting: classical computations
- Classical security: classical queries
- Cipher broken by adversary with
  - data $\ll 2^n$
  - time $\ll 2^k$
  - success $> 3/4$



Security notions: Quantum Q1

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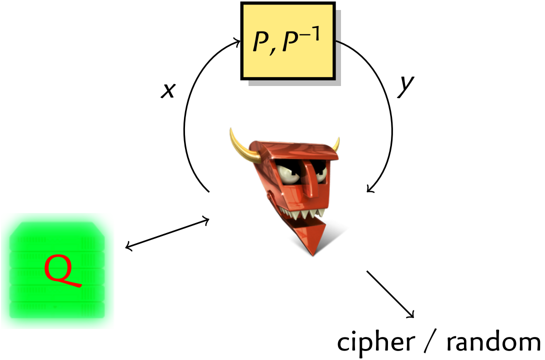

Security notions: Quantum Q2

- PRF security: given access to P/P^{-1}, distinguishing E from random
- Quantum setting: quantum computations
- Quantum security: quantum (superposition) queries
- Cipher broken by adversary with
  - data $\ll 2^n$
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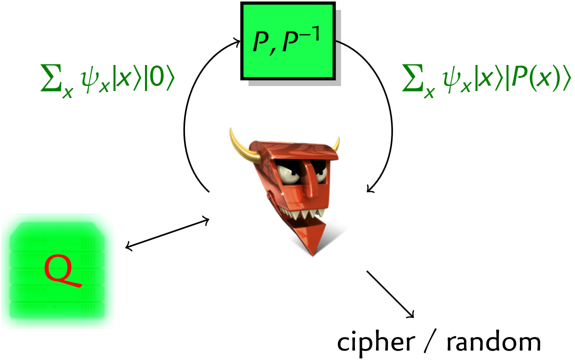
  - success $> 3/4$



About the models

Q1 model: classical queries

- Build a quantum circuit from classical values
- Example: breaking RSA with Shor's algorithm

Q2 model: superposition queries

- Access quantum circuit implementing the primitive with a secret key
- Example: breaking CBC-MAC with Simon's algorithm

- The Q2 model is very strong for the adversary
  - Simple and clean generalisation of classical oracle
  - Aim for security in the strongest (non-trivial) model
  - A Q2-secure block cipher is useful for security proofs of modes

Outline

Grover's algorithm

- Search for a marked element in a set X
- Set of marked elements M, with $|M| \geq \varepsilon \cdot |X|$

Classical algorithm

1: **loop**
2: $x \leftarrow$ SETUP() ▷ Pick a random element in X, cost S
3: **if** CHECK(x) **then** ▷ Check if it is marked, cost C
4: **return** x

- $1/\varepsilon$ repetitions expected
- Complexity $(S + C)/\varepsilon$

Grover's algorithm

- Search for a marked element in a set X
- Set of marked elements M, with $|M| \geq \varepsilon \cdot |X|$

Grover Algorithm (as a quantum walk)

Quantum algorithm to find a marked element using:

- SETUP: builds a uniform superposition of inputs in X
- CHECK: applies a control-phase gate to the marked elements

- Only $1/\sqrt{\varepsilon}$ repetitions needed
- Complexity $(S + C)/\sqrt{\varepsilon}$

- Can produce a uniform superposition of M
- Can provide an oracle without measuring (nesting)
- Variant to measure ε (quantum counting)

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Brute-force attack

- We can use Grover's algorithm for a quantum brute-force key search

1. Capture a few known plaintext/ciphertext: $C_i = E_{\kappa^*}(P_i)$
2. SETUP: builds a uniform superposition of $\{0,1\}^k$ — $S = 1$
3. CHECK(κ): test whether $C_i = E_\kappa(P_i)$ — $\varepsilon = 2^{-k}, C = 1$

- Complexity $O(2^{k/2})$
  - Quadratic gain
- Uses the Q1 model
  - Classical data (C_i, P_i)
  - Quantum circuit independant of the secret key κ^*

Outline

Introduction
- Quantum Computing

Brute-force
- Grover's algorithm

Differential
- Distinguisher
- Last-round attack

Truncated differential
- Distinguisher
- Last-round attack

Conclusion

Differential distinguisher: classical

- Assume a *differential* $\delta_{in}, \delta_{out}$ given, with

$$h := -\log \Pr_x[E(x \oplus \delta_{in}) = E(x) \oplus \delta_{out}] \ll n,$$

Classical algorithm: search for right pairs

1: **for** $0 \leq i < 2^h$ **do**
2: $x \leftarrow$ Rand()
3: **if** $E(x \oplus \delta_{in}) = E(x) \oplus \delta_{out}$ **then**
4: **return** `cipher`
5: **return** `random`

- Complexity $O(2^h)$

Differential distinguisher: quantum

- Assume a *differential* $\delta_{\text{in}}, \delta_{\text{out}}$ given, with

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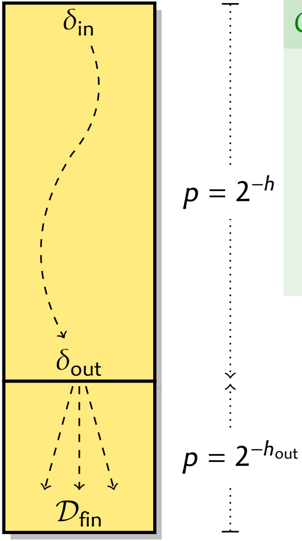
Quantum algorithm: Grover search for right pair

1. SETUP: builds a uniform superposition of $\{0,1\}^n$ — $S = 1$
2. CHECK(x): test whether $E(x \oplus \delta_{\text{in}}) = E(x) \oplus \delta_{\text{out}}$ — $\varepsilon = 2^{-h}, C = 1$

- Complexity $O(2^{h/2})$
  - Quadratic gain
- Uses the Q2 model
  - Superposition queries to E with secret key

Last-Round attack: classical

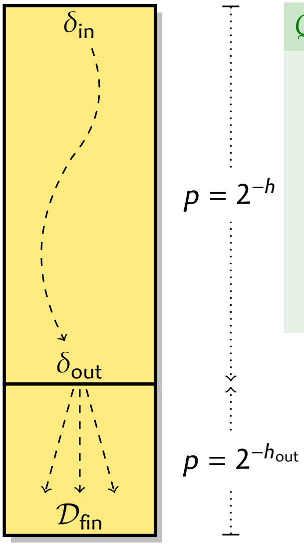


Classical algorithm

for $0 \leq i < 2^h$ **do**
  $x \leftarrow$ RAND()
  ▷ Filter possible output differences
  if $E(x) \oplus E(x \oplus \delta_{in}) \in \mathcal{D}_{fin}$ **then**
    Find last key candidates for $(x, x \oplus \delta_{in})$
    Try all possibilities for remaining key bits

- Finding partial key candidates costs $C_{k_{out}}$
  - Between 1 and $2^{k_{out}}$
- $T = 2^h + 2^{h-n+\Delta_{fin}} \cdot \left(C_{k_{out}} + 2^{k-h_{out}}\right)$

Last-Round attack: quantum Q2

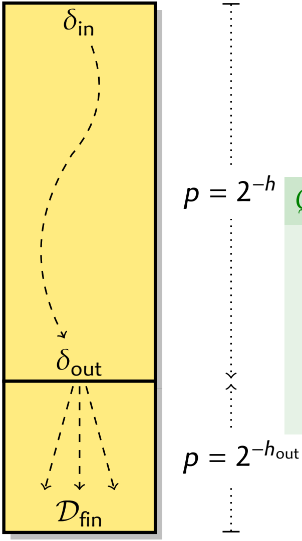


Quantum algorithm: Grover search for right pair

1. SETUP: builds a uniform superposition of $X = \{x : E(x) \oplus E(x \oplus \delta_{in}) \in \mathcal{D}_{fin}\}$ using nested Grover algorithm $S = 2^{(n-\Delta_{fin})/2}$
2. CHECK(x): Find last key cand. for $(x, x \oplus \delta_{in})$ Run nested Grover over remaining key bits $\varepsilon = 2^{n-h-\Delta_{fin}}, C = C^*_{k_{out}} + 2^{(k-h_{out})/2}$

- Repeat key recovery with right pair
- Finding partial key candidates costs $C^*_{k_{out}}$
  - Between 1 and $2^{k_{out}/2}$
- $T = 2^{h/2} + 2^{(h-n+\Delta_{fin})/2} \cdot \left(C^*_{k_{out}} + 2^{(k-h_{out})/2}\right)$

Last-Round attack: quantum Q1



- Previous attack uses superposition queries
- Alternatively, make 2^h classical queries
  - Interesting if $2^h < 2^{k/2}$
  - E.g. AES-256

Quantum algorithm: Grover search for right pair

1. SETUP: builds superposition of classical data using quantum memory $S = 1$
2. CHECK(x): same as Q2

$$\varepsilon = 2^{n-h-\Delta_{\text{fin}}}, C = C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2}$$

- $T = 2^h + 2^{(h-n+\Delta_{\text{fin}})/2} \cdot \left(C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2}\right)$

Outline

Truncated differential distinguisher: classical

- Assume vector spaces $\mathcal{D}_{\text{in}}, \mathcal{D}_{\text{out}}$ given (dim. $\Delta_{\text{in}}, \Delta_{\text{out}}$), with

$$h := -\log \Pr_{x,\delta \in \mathcal{D}_{\text{in}}} [E(x \oplus \delta) \oplus E(x) \in \mathcal{D}_{\text{out}}] \ll n - \Delta_{\text{out}},$$

Classical algorithm (using structures)

1: **for** $0 \leq i < 2^{h-2\Delta_{\text{in}}}$ **do**
2: $\quad x \leftarrow \text{Rand}()$
3: $\quad L \leftarrow \{E(x \oplus \delta) : \delta \in \mathcal{D}_{\text{in}}\}$
4: $\quad$ **if** $\exists y_1, y_2 \in L$ s.t. $y_1 \oplus y_2 \in \mathcal{D}_{\text{out}}$ **then**
5: $\qquad$ **return** `cipher`
6: **return** `random`

- Complexity $O(2^{h-\Delta_{\text{in}}})$

Truncated differential distinguisher: quantum

- Assume vector spaces $\mathcal{D}_{\text{in}}, \mathcal{D}_{\text{out}}$ given (dim. $\Delta_{\text{in}}, \Delta_{\text{out}}$), with

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Quantum algorithm: Grover search for structure with right pair

1. SETUP: builds a uniform superposition of $\{0,1\}^n$ $S = 1$
2. CHECK(x): test whether $\exists y_1, y_2 \in x \oplus \mathcal{D}_{\text{in}}$ s.t. $y_1 \oplus y_2 \in \mathcal{D}_{\text{out}}$
$\varepsilon = 2^{-h+2\Delta_{\text{in}}}, C = ?$

Finding collisions

- Fiding $y_1, y_2 \in L$ s.t. $y_1 \oplus y_2 \in \mathcal{D}_{\text{out}}$: truncate and find collisions

Classical algorithm

```
SORT(L)
for 0 < i < |L| do
    if L[i] = L[i+1] then return L[i]
return ⊥
```

- Complexity $\tilde{O}(N)$

Quantum algorithmic: Ambainis' element distinctness

- Quantum walk algorithm to find collisions
- Complexity $O(N^{2/3})$ — less than quadratic speedup!
- Uses memory $O(N^{2/3})$

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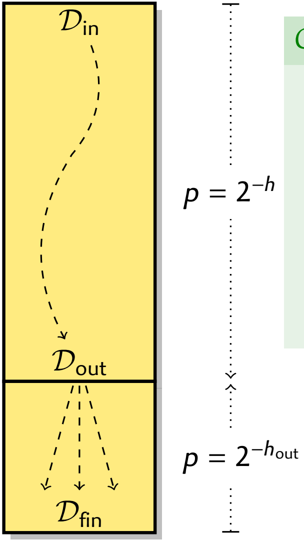
- Complexity $O(2^{h/2-\Delta_{\text{in}}/3})$ — less than quadratic speedup
- Uses the Q2 model
  - Superposition queries to E with secret key

Last-Round attack: classical



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  ▷ Filter possible output differences
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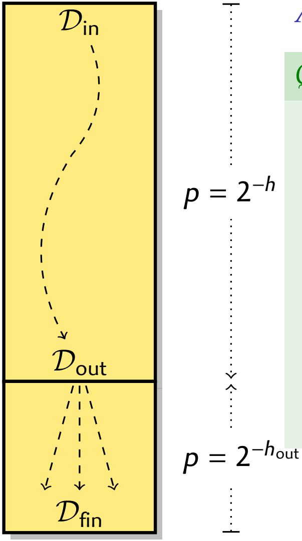
Last-Round attack: quantum Q2



Assume each structure has pairs with difference in $\mathcal{D}_{fin}$

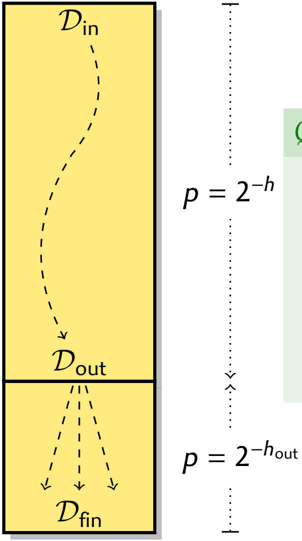
Q2 algo: Grover search for structure with right pair

1. SETUP: unif. superposition $S = 1, \varepsilon = 2^{2\Delta_{\text{in}} - h}$
2. CHECK(x): Grover search over pairs in $x \oplus \mathcal{D}_{\text{in}}$
   1. SETUP: Ambainis to find pairs with output in $\mathcal{D}_{\text{fin}}$ $S' = 2^{(n-\Delta_{\text{fin}})/3}$
   2. CHECK(x_1, x_2): Find last key candidates Run nested Grover over remaining key bits, $\varepsilon' = 2^{-2\Delta_{\text{in}}+(n-\Delta_{\text{fin}})}, C' = C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2}$

$$C = 2^{\Delta_{\text{in}}-(n-\Delta_{\text{fin}})/6} + 2^{\Delta_{\text{in}}+(\Delta_{\text{fin}}-n)/2} \left(C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2} \right)$$

- $T = 2^{h/2-(n-\Delta_{\text{fin}})/6} + 2^{(h-n+\Delta_{\text{fin}})/2} \cdot \left(C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2} \right)$

Last-Round attack: quantum Q1



- Alternatively, use classical queries
- Filter pairs with output in $\mathcal{D}_{\text{fin}}$ classically

Q1 algo: Grover search for structure with right pair

1. SETUP: builds superposition of classical data using quantum memory $S = 1$
2. CHECK(x_1, x_2): Find last key candidates
   Run nested Grover over remaining key bits
   $\varepsilon = 2^{n-h-\Delta_{\text{fin}}}, C = C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2}$

- $T = 2^{h-\Delta_{\text{in}}} + 2^{(h-n+\Delta_{\text{fin}})/2} \cdot \left(C^*_{k_{\text{out}}} + 2^{(k-h_{\text{out}})/2}\right)$

Summary: simplified complexities

- Simple differential distinguisher

$$D_C = 2^h \qquad D_{Q1} = 2^h = D_C \qquad D_{Q2} = 2^{h/2} = \sqrt{D_C}$$
$$T_C = 2^h \qquad T_{Q1} = 2^h = T_C \qquad T_{Q2} = 2^{h/2} = \sqrt{T_C}$$

- Simple differential LR attack

$$D_C = 2^h \qquad D_{Q1} = 2^h = D_C \qquad D_{Q2} = 2^{h/2} = \sqrt{D_C}$$
$$T_C = 2^h + C_k \qquad T_{Q1} = 2^h + C_k^* \qquad T_{Q2} = 2^{h/2} + C_k^* \approx \sqrt{T_C}$$

- Truncated differential distinguisher

$$D_C = 2^{h-\Delta_{in}} \qquad D_{Q1} = 2^{h-\Delta_{in}} = D_C \qquad D_{Q2} = 2^{h/2-\Delta_{in}/3} > \sqrt{D_C}$$
$$T_C = 2^{h-\Delta_{in}} \qquad T_{Q1} = 2^{h-\Delta_{in}} = T_C \qquad T_{Q2} = 2^{h/2-\Delta_{in}/3} > \sqrt{T_C}$$

- Truncated differential LR attack *Assuming > 1 filtered pairs / structure*

$$D_C = 2^{h-\Delta_{in}} \qquad D_{Q1} = 2^{h-\Delta_{in}} = D_C \qquad D_{Q2} = 2^{h/2-(n-\Delta_{fin})/6} > \sqrt{D_C}$$
$$T_C = 2^{h-\Delta_{in}} + C_k \qquad T_{Q1} = 2^{h-\Delta_{in}} + C_k^* \qquad T_{Q2} = 2^{h/2-(n-\Delta_{fin})/6} + C_k^* > \sqrt{T_C}$$

Concrete examples

- Truncated differential attacks have less than quadratic speedup
- Can become worse than Grover key search (not an attack)
- The best quantum attack is not always
  a quantum version of the best classical attack

LAC (reduced LBlock, $n = 64$)

- Differential with probability $2^{-61.5}$
  - Classical distinguisher with complexity $2^{62.5}$
  - Quantum distinguisher with complexity $2^{31.75}$
- Truncated differential with $\Delta_{\text{in}} = 12, \Delta_{\text{out}} = 20, 2^h = 2^{-44} + 2^{-55.3}$
  - Classical distinguisher with complexity $2^{60.9}$
  - Quantum distinguisher with complexity $2^{33.4}$

Concrete examples

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KLEIN-64 ($n = 64$)

- Truncated differential with $h = 69.5$, $\Delta_{in} = 16$, $\Delta_{fin} = 32$, $k = 64$, $k_{out} = 32$, $h_{out} = 45$
  - Classical attack with complexity $2^{58.2}$
  - Quantum attack with complexity $> 2^{32}$

Concrete examples

- Truncated differential attacks have less than quadratic speedup
- Can become worse than Grover key search (not an attack)
- The best quantum attack is not always
  a quantum version of the best classical attack

KLEIN-96 ($n = 64$)

- Truncated differential with $h = 78$, $\Delta_{in} = 32$, $\Delta_{fin} = 32$, $k = 96$, $k_{out} = 48$, $h_{out} = 52$
  - Classical attack with complexity 2^{90}
  - Q2 attack with complexity $2^{47.3}$
  - Q1 attack with complexity $2^{47.96}$

Conclusions

- We fixed some mistakes from the ToSC version
  - Updated version on arXiv:1510.05836

- Quantification of classical attacks using Grover and Ambainis
  - Differential, truncated differential and linear cryptanalysis

- “It’s complicated”
- Up to quadratic speedup
  - If key search is the best classical attack,
    Grover key search is the best quantum attack
- Data complexity can only be reduced using quantum queries
- Cipher with $k > n$ are most likely to see quadratic speedup
  - Attacks with classical queries (Q1 model) possible

Bonus slide: Linear cryptanalysis

- Linear distinguisher

$$D_C = 1/\varepsilon^2 \qquad D_{Q1} = 1/\varepsilon^2 = D_C \qquad D_{Q2} = 1/\varepsilon = \sqrt{D_C}$$
$$T_C = 1/\varepsilon^2 \qquad T_{Q1} = 1/\varepsilon^2 = T_C \qquad T_{Q2} = 1/\varepsilon = \sqrt{T_C}$$

- Linear attack with ℓ r-round distinguishers (Matsui 1)

$$D_C = 1/\varepsilon^2 \qquad D_{Q1} = \ell/\varepsilon^2 > D_C \qquad D_{Q2} = \ell/\varepsilon > \sqrt{D_C}$$
$$T_C = \ell/\varepsilon^2 + 2^{k-\ell} \qquad T_{Q1} = \ell/\varepsilon^2 + 2^{(k-\ell)/2} \qquad T_{Q2} = \ell/\varepsilon + 2^{(k-\ell)/2} > \sqrt{T_C}$$

- Last-round linear attack (Matsui 2)

$$D_C = 1/\varepsilon^2 \qquad D_{Q1} = 1/\varepsilon^2 = D_C \qquad D_{Q2} = 2^{k_{\text{out}}/2}/\varepsilon > \sqrt{D_C}$$
$$T_C = C_k \qquad T_{Q1} = 1/\varepsilon^2 + \sqrt{C_k} \qquad T_{Q2} = \sqrt{C_k} = \sqrt{T_C}$$